Convergence of Numerical Solutions of Open-Ended Waveguide by Modal Analysis and Hybrid Modal–Spectral Techniques

JOSE A. ENCINAR AND JESÚS M. REBOLLAR

Abstract —Two different methods are considered to deal with open-ended waveguides with an infinite metallic flange. The first one, modal analysis, is valid only when the aperture is radiating into a lossy medium. The second one is based on a hybrid modal–spectral technique, and it is valid for any medium, with or without losses.

The rectangular aperture problem is solved by both methods, and the influence of different parameters on the convergence of numerical solutions is studied for each method.

Finally, a comparison between both methods is presented for lossy and low-loss media.

I. Introduction

OPEN-ENDED waveguides are very useful for many applications such as feed for reflector antennas, flush-mounted antenna for space craft, or thermography and hyperthermia applicators.

The aperture problem has been solved by different approximate methods. Most of them consider only the fundamental mode in the aperture [1]. Others [2]–[4] consider the fundamental and higher modes in the aperture in order to solve the problem more accurately.

When the open-ended waveguide is radiating into lossy media such as biological tissues or plasma, the model proposed in [4] and [5] will be considered here. In this model, the lossy medium is assumed to be inside a large cross-section imaginary waveguide. The electromagnetic fields are concentrated near the aperture due to dielectric losses. Therefore, the effects produced by metallic walls of the large imaginary waveguide are negligible. With this assumption, the aperture problem is reduced to a problem of discontinuity between two different waveguides with cross-sections S_i and S_o, being $S_o \gg S_i$ (Fig. 1). Then the problem can be solved by modal analysis [6], [7], assuming that the electromagnetic fields are expressed as a superposition of the eigenmodes in each region.

This model is not valid for low-loss or lossless media. However, the angular spectrum of plane waves can be used to represent the electromagnetic field in half-space in any kind of medium [8].

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According to the above consideration, we propose a second method that uses eigenmodes in the waveguide and angular plane-wave spectrum in half-space (see Fig. 2). Then this method will be valid for any medium with or without losses, and it will be called the hybrid modal–spectral method (HMSM).

Although the analysis is exact for both methods, there are some difficulties in the numerical resolution of the aperture problem.

First, in the modal analysis, an imaginary waveguide of a large cross section tries to simulate the open space for lossy media, and the imaginary waveguide dimensions must increase when the losses decrease. Second, the number of modes retained in each waveguide is finite. Finally, when the problem is solved by the HMSM, the integration in half-space is performed numerically with a finite number of samples. These factors, i.e., imaginary waveguide dimensions, number of modes, and number of samples, influence on the numerical solution of the aperture problem and the convergence of the results must be studied.

Some results have been published about the influence of the ratio "number of modes/cross section" for each waveguide on the numerical solutions of discontinuities found by modal analysis [7], [9], but in these works the two waveguides have a similar cross section. In this paper, the imaginary output waveguide simulates the open space, so its cross section is very large (about 100 times the input waveguide cross section). This leads to considering a very large number of modes in the output waveguide (from 400 to 6400) in order to solve the problem accurately, and the ratio between the number of modes in each waveguide M/N is more critical in order to save computer time and enhance accuracy.

The aperture is characterized by the reflection matrix S_{11}, where the coefficient $S_{11}(i, j)$ is the amplitude of the reflected mode for jth incident mode, and then $S_{11}(1,1)$ is the reflection coefficient for the fundamental mode (TE_{10}). The reflection matrix contains not only the reflection characteristics of the fundamental mode, but also those of the higher order evanescent modes. This matrix is calculated by both methods and the results are then compared.

In the previous works, the convergence has been studied by comparing the TE_{10} characteristics only. In this paper, the convergence of the entire reflection matrix S_{11} is studied.

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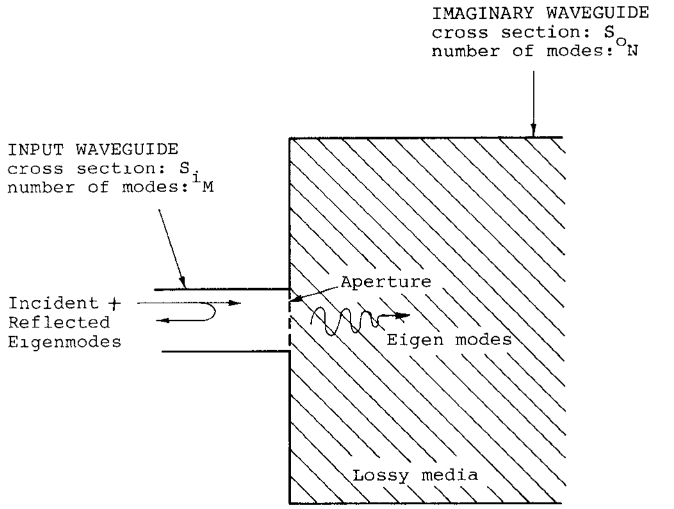

Fig. 1 Proposed model for an aperture radiating in a lossy medium.

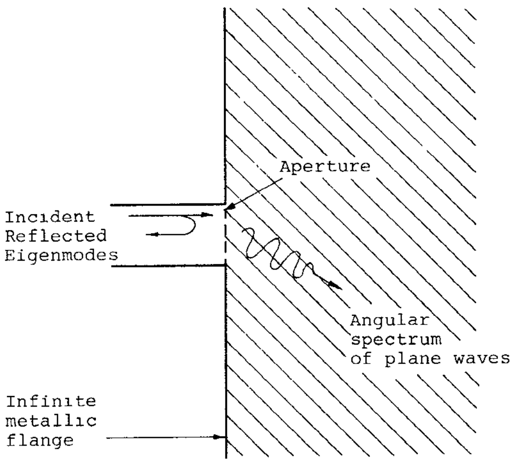

Fig. 2. Aperture radiating into half-space.

The electromagnetic fields at the aperture for any incident field in the waveguide are easily calculated from the reflection matrix S_{11}. When the TE_{10} mode is incident, the aperture field obtained by both methods is plotted and then compared.

The reflection matrix S_{11} characterizes the aperture and allows one to join the aperture problem with other waveguides discontinuities described by their scattering matrices, as has been proposed in [10]. The scattering matrix of each discontinuity is obtained by means of modal analysis. So, modal analysis and the HMSM can be applied together to design and optimize circuit components with multiple transverse discontinuities, such as adapters, corrugated polarizers, horns, etc., including the radiating aperture.

Finally, the radiation characteristics of the complete structure can be obtained with the proposed techniques.

II. Modal Analysis

Modal analysis [6], [7] is a standard computer-oriented method for solving discontinuity problems in waveguides.

The aperture is considered as a discontinuity between two waveguides of very different cross section (see Fig. 1),

TABLE I
TE_{10} Reflection Coefficient (Magnitude $\|\rho\|$, Phase ϕ) of a Square Aperture ($A \times B = 19 \times 19$ mm) for Different Lossy Media

	Number of Modes		Imaginary waveguide dimensions 5A x 5B		10A x 10B		20A x 20B	
	M	N	$\|\rho\|$	ϕ	$\|\rho\|$	ϕ	$\|\rho\|$	ϕ
ϵ_r = 46-j13	16	400	.2076	171.9	.1907	172.0		
	16	1600	.2092	171.9	.2077	171.8		
ϵ_r = 5.8-j0.6	16	400	.7238	39.7	.6735	34.1	.6721	29.2
	16	1600	.7255	40.3	.6793	36.2	.6776	35.1
	16	6400	.7258	40.4	.6812	36.9	.6834	37.2
ϵ_r= 5.8-j0.06	16	400	.7673	38.3	.6066	28.7	.6756	25.4
	16	1600	.7688	38.9	.6137	30.9	.6813	31.4
	16	6400	.7691	39.0	.6160	31.6	.6874	33.5

Frequency = 2 GHz. Permittivity in the waveguide $\epsilon_r = 30$.

and the electromagnetic fields are expressed as a superposition of the incident and reflected modes in each waveguide

$$\vec{E}^{I,O} = \sum_i \left(d_i^{I,O} + a_i^{I,O}\right)\vec{e}_i^{I,O} \tag{1}$$

$$\vec{H}^{I,O} = \sum_i \left(d_i^{I,O} - a_i^{I,O}\right)\vec{h}_i^{I,O} \tag{2}$$

where I,O means input and output waveguides, respectively, $\vec{e}_i, \vec{h}_i$ are the electric and magnetic fields for the i mode, and d_i, a_i are the amplitudes of the incident and reflected i mode.

The continuity condition on transverse components of the electromagnetic fields is applied at the discontinuity plane, and it leads to obtaining the whole scattering matrix of the aperture S. The scattering submatrix S_{11} is the reflection matrix mentioned before.

A. Numerical Results

A rectangular aperture of cross section $S_i = A \times B$ radiating into different biological lossy media is considered. TE^x family modes are used to represent the fields in each region.

Sixteen modes have been retained in the input waveguide ($M = 16$). It has been proved that this number of modes is enough to represent the electromagnetic field in the considered input waveguide ($A = 19$ mm, $B = 19$ mm).

The problem has been solved considering different number of modes in the imaginary output waveguide ($N = 400, 1600, 6400$), and also different imaginary waveguide cross sections ($S_o = 25, 100, 400\ S_i$).

The reflection coefficient for the TE_{10} mode $S_{11}(1,1)$, versus the parameters N and S_o, is shown in Table I for three different lossy media. Note that the ratio $(M/N)/(S_i/S_o)$ is maintained fixed in each diagonal.

From this table, it can be observed that the reflection coefficient for the fundamental mode TE_{10} is not very dependent on the number of modes used in the output waveguide. Also, the imaginary waveguide dimensions necessary to simulate the open space can be obtained for each lossy media. When $S_o \geqslant 25 S_i$ (skin or muscle) or $S_o \geqslant 100 S_i$ (fat or bone), the effect produced by metallic walls of the

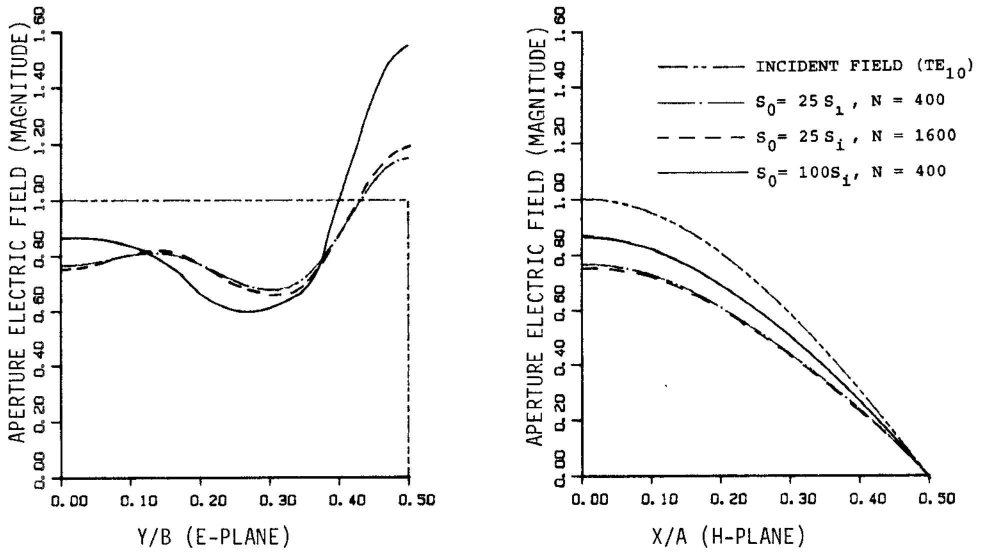

Fig. 3. Magnitude of the aperture electric field for different values of N and S_o. Frequency = 2 GHz. Permittivity of the dielectric filling the input waveguide, $\epsilon_r = 30$; filling the half-space, $\epsilon_r = 46 - j13$. $A \times B = 19 \times 19$ mm.

imaginary waveguide is negligible, and the model is valid.

From the comparison of the other reflection matrix coefficients $S_{11}(i, j)$ for different values of the parameters N and S_o, the following conclusions can be observed.

1) The reflection matrix coefficients $S_{11}(i, j)$ for higher modes are very influenced by the number of modes M and N used in each waveguide, and the higher the order of the mode the greater the influence.

2) If we change the cross sections S_i and S_o and the number of modes M and N, but maintaining the ratio $(M/N)/(S_i/S_o)$ fixed, the reflection matrix will not change considerably.

3) The coefficients $S_{11}(i, j)$, when $i \gg 1$ and $j \gg 1$, converge to different values if the ratio $(M/N)/(S_i/S_o)$ is changed.

4) If $(M/N) > (S_i/S_o)$, the calculated reflection coefficient for the higher order modes considered $(i \gg 1,\ j \gg 1)$ is $S_{11}(i, j) \simeq 1$. This results can be explained by the fact that for theses modes the continuity condition at the aperture cannot be satisfied with the number of modes retained in the output waveguide.

The magnitude of the transverse electric field at the aperture is plotted for the four cases shown in Table I(a), see Fig. 3. For the two cases when $(M/N) = (S_i/S_o)$, the plottings are equal. For $(M/N) < (S_i/S_o)$, the aperture field is very similar to that of $(M/N) = (S_i/S_o)$. Finally, when $(M/N) > (S_i/S_o)$, the obtained aperture field is different from the other cases.

These results can be summarized in the following conclusions.

1) When $(M/N) \leqslant (S_i/S_o)$, the problem is accurately solved and the coefficients $S_{11}(i, j)$, $i \gg 1$, $j \gg 1$, do not have a great influence on the representation of the aperture fields.

2) In order to solve the problem efficiently, the optimum ratio is $(M/N) = (S_i/S_o)$. This is the only ratio that also allows us to solve accurately the problem when the direction of the incident field is reversed. This optimum ratio agrees with that presented in [9].

TABLE II
VSWR AND PHASE ANGLE (ϕ) OF THE TE_{10} REFLECTION COEFFICIENT FOR WR-90 WAVEGUIDE OPERATED AT 10 GHz

Reference [11] VSWR ϕ	Our Results ($A_2=10A_1$, $B_2=10B_1$) VSWR ϕ	
2.7 -179.5	2.66 -175.1	16x2025 modes
	2.53 -174.9	36x900 modes

TABLE III
MAGNITUDE OF REFLECTION COEFFICIENT FOR WR-90 WAVEGUIDE IN CONTACT WITH DIFFERENT MEDIA

f=9 GHz. WR-90	Experimental (5) $\lvert\rho\rvert^2$	Modal Analysis $A_2=10\ A_1$, $B_2=10\ B_1$ M=36, N=900 $\lvert\rho\rvert^2$
water	0.72	0.72
chloroform	0.23	0.24

B. Experimental and Other Numerical Results

Numerical results of VSWR and phase angle of the reflected fundamental mode for the particular case of a WR-90 waveguide (X-band, 22.86 by 10.16 mm) operating at 10 GHz are presented in [11].

A comparison is made for a lossy material with relative dielectric permittivity $\epsilon_r = 4.5 - j0.9$ (see Table II)

In [5], experimental and numerical results are presented for the magnitude of reflection coefficient for a WR-90 waveguide operating at 9 GHz in contact with water ($\epsilon_r = 64 - j30.5$) and chloroform ($\epsilon_r = 4.49 - j0.85$). A comparison between experimental results and our results is presented in Table III and a good agreement is observed.

III. Hybrid Modal–Spectral Method (HMSM)

In this method, the electromagnetic fields in the waveguide are expressed in the same way than in modal analysis, and by angular spectrum of plane waves [8] in half-space

$$\vec{E}_{st}(x, y, z) = \iint_{-\infty}^{\infty} \vec{F}(k_x, k_y) e^{-j(k_x x + k_y y + k_z z)} dk_x dk_y \tag{3}$$

where the angular plane-wave spectrum $\vec{F}(k_x, k_y)$ and the transverse electric field at the aperture $\vec{E}_{st}(x, y, 0)$ are a pair of two-dimensional Fourier transforms

$$\vec{F}(k_x, k_y) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \vec{E}_{st}(x, y, 0) e^{j(k_x x + k_y y)} dx dy. \tag{4}$$

The continuity condition of the transverse magnetic field at the aperture plane is considered and it leads to the expression

$$(T+Q)A = (T-Q)D \tag{5}$$

where $D = (d_i)$ and $A = (a_i)$ are column vectors, T is a diagonal matrix, whose coefficients are

$$T_{ii} = \iint_{\text{Aperture}} (\vec{e}_i^* \wedge \vec{h}_i) \cdot \vec{dS} \tag{6}$$

and using TE^x family modes, the Q matrix coefficients are

$$Q_{i,j} = \frac{(2\pi)^2}{\eta} \iint_{-\infty}^{\infty} \frac{(k^2 - k_x^2)}{k k_z} \cdot F_{yi}(k_x, k_y) F_{yj}^*(k_x, k_y) dk_x dk_y. \tag{7}$$

Solving these equations, we obtain the amplitudes of the reflected modes A, for any incident field D

$$A = \left[(T+Q)^{-1}(T-Q)\right] D \tag{8}$$

and

$$S_{11} = (T+Q)^{-1}(T-Q) \tag{9}$$

is the reflection matrix of the aperture.

To calculate the coefficients Q_{ij}, the variables are converted into spacial domain, using Parseval's theorem, in order to avoid the integration for infinite limits.

For a rectangular aperture of dimensions $A \times B$, the coefficients become

$$Q_{ij} = \frac{1}{(2\pi)} \frac{1}{\eta} \int_{-A}^{A} \int_{-B}^{B} h_0^{(2)}\left(k\sqrt{x^2 + y^2}\right) \cdot \left(k^2 + \frac{d}{dx^2}\right) R_{i,j}(x, y) dx dy \tag{10}$$

where $h_0^{(2)}(k\sqrt{x^2 + y^2})$ is the spherical Hankel function of order zero and the second kind, and $R_{i,j}(x, y)$ is the correlation function of the tangential electric field of the i mode with the j mode, and for the TE^x family modes the function is

$$R_{i,j}^{yy}(x, y) = \iint_{-\infty}^{\infty} e_{yi}(x', y')\, e_{yj}(x' + x, y' + y)\, dx' dy'. \tag{11}$$

Finally, the function $h_0^{(2)}(k\sqrt{x^2 + y^2})$, $0 < x \leqslant A$ and $0 < y \leqslant B$, is expanded into two-dimensional Fourier series and then the integrals can be solved analytically.

A very similar method, called the correlation matrix method, was proposed by R. H. MacPhie *et al.* [3].

In this method, the following parameters will have an influence on the numerical solution of the number of modes retained in the waveguide, and the number of samples taken from $h_0^{(2)}(k\sqrt{x^2 + y^2})$ for the Fourier expansion in the aperture.

A. Numerical Results

The reflection matrix for several rectangular apertures has been obtained by the HMSM. The problem was solved with a different number of samples taken from $h_0^{(2)}(k\sqrt{x^2 + y^2})$.

In Table IV, the reflection coefficient of the square aperture already studied by the modal analysis is shown. Looking at this table, we are able to make the following comments.

1) The convergence of the reflection coefficient $S_{11}(1,1)$ is obtained for a small number of samples (33×33 or 16×16).

TABLE IV
TE$_{10}$ Reflection Coefficient (Magnitude $|\rho|$, Phase ϕ) of a Square Aperture ($A \times B = 19 \times 19$ mm) for Different Lossy Media

Number of Samples	Muscle ϵ_r=46-j13 $\lvert\rho\rvert$	ϕ	fat ϵ_r=5.8-j.6 $\lvert\rho\rvert$	ϕ	ϵ_r=5.8-j.06 $\lvert\rho\rvert$	ϕ
9	.187	145.3	.6734	28.9	.696	26.3
17	.199	159.7	.6819	33.4	.707	30.8
33	.204	165.9	.6856	35.7	.712	33.1
65	.207	168.9	.6872	36.9	.714	34.3
129	.208	170.3	.6878	37.5	.715	34.9

Frequency = 2 GHz. $\epsilon_r = 30$ in the waveguide.

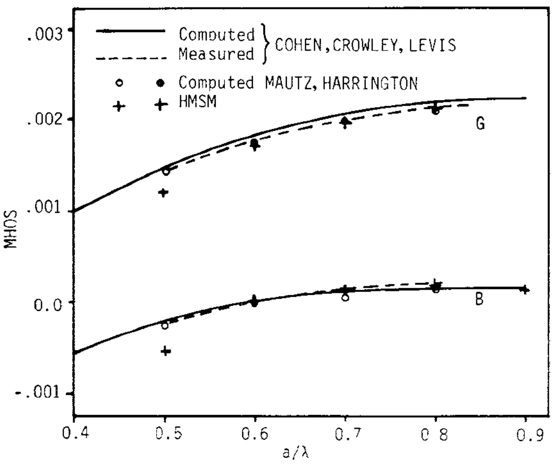

Fig. 4. Equivalent aperture admittance for a square aperture of width a obtained by different authors [2].

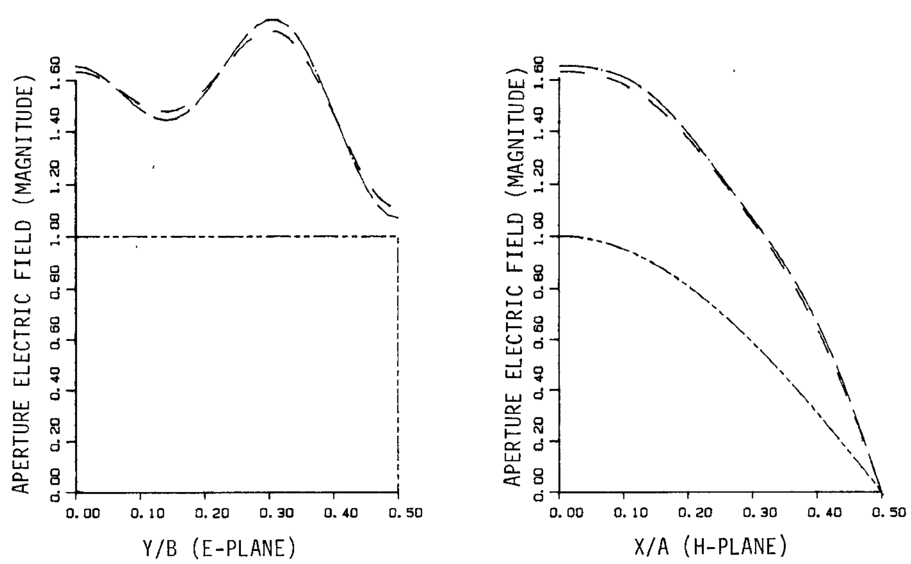

Fig. 5. Magnitude of the aperture electric field obtained by modal analysis-·-; HMSM ---. $A \times B = 19 \times 19$ mm. Frequency = 2 GHz. $\epsilon_r = 30$ in the waveguide, $\epsilon_r = 5.8 - j0.6$ in half-space.

2) The convergence is slower, mainly for the phase, for muscle or skin ($\epsilon_r = 46.j13$), compared to fat or bone ($\epsilon_r = 5.8 - j0.6$). For the muscle medium, due to the high permittivity, the electrical dimensions of the aperture are greater, and therefore a higher number of samples (33×33) should be used at the aperture.

Other reflection matrix coefficients do not change considerably when the number of samples is greater than 32×32. In that case, there is no difference between the plots of the aperture fields obtained with a different number of samples. Also, these plots coincide with the ones obtained by the modal analysis $(M/N) \leqslant (S_t/S_o)$, Fig. 3.

Finally, the input admittance of the aperture has been calculated and compared with that obtained by other authors [2], [12]. Fig. 4 shows the comparison for a square aperture radiating in free space. Good agreement is obtained with the experimental and other numerical results.

IV. Comparison and Discussion

The formulation of the problem is very similar by both methods. The main difference between them consists of the electromagnetic field representation in half-space.

In the modal analysis, the electromagnetic fields are assumed to be a summation of the eigenmodes of an imaginary waveguide with a large cross section, and in the spectral method the summation is substituted by a continuous integral. But, the continuous integral is sampled in the aperture in order to solve the problem numerically.

It must be remarked that the number of modes in the imaginary waveguide depends on the output waveguide dimensions and consequently on the losses in the medium. The lower the losses, the higher the imaginary waveguide dimensions and consequently a higher number of modes should be considered. However, in the HMSM, the number of samples depends only on the aperture dimensions and the permittivity of the medium but not on the losses in the medium.

Comparing the results obtained by both methods, the following comments can be made.

1) The effect of increasing the number of samples in Table IV is similar to increasing the number of modes in the imaginary waveguide in Table I.

2) The reflection coefficients of the fundamental modes obtained by both methods are very similar for lossy media. But for the third case considered, very low-loss media, the modal analysis solutions do not converge to the ones obtained by the HMSH, even when the cross section of imaginary waveguide is 400 times that of the input waveguide. In such a case, the effect of the lateral walls of the output waveguide cannot be neglected and the problem must be solved by the HMSM.

3) Other reflection matrix coefficients $S_{11}(i, j)$ obtained by the HMSM are comparable to the ones obtained by the modal analysis when $(M/N) \leqslant (S_t/S_o)$ and the imaginary waveguide is large enough for a lossy media.

Finally, the magnitude of the transverse electric field calculated at the aperture by both methods is compared. Good agreement is observed for biological tissues (see Fig. 5 for fat media).

V. Conclusions

The reflection matrix and the aperture field of a rectangular open-ended waveguide with flange are calculated by both proposed methods.

When the aperture is radiating into a lossy medium, the problem can be solved by modal analysis using the imaginary output waveguide model. The dimensions of the imaginary waveguide depends on the losses of the medium; the lower the losses, the greater the output waveguide cross section necessary to simulate the half-space.

In the modal analysis, to solve the problem accurately, an optimum ratio "number of modes/cross section" for the output waveguide has been obtained, $N/S_o = M/S_t$.

Then the number of modes in the output waveguide N must increase in the same proportion as S_o.

The same problems have been solved by the HMSM, and the agreement in the results prove the validity of the modal analysis for lossy media.

For low-loss media, the imaginary waveguide cross section S_o must be very large and a great number of modes N should be considered in the imaginary waveguide. In that case, the problem must be solved by the HMSM in order to increase accuracy and to save computer time.

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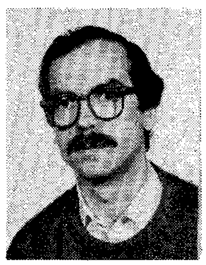

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